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CONSISTENT FINITE DIFFERENCE EQUATIONS FOR THIN ELASTIC DISKS OF VARIABLE THICKNESS*

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Abstract—The paper is concerned with the establishment of consistent finite difference equations for the displacement field of a thin elastic disk of variable thickness under given body forces and edge tractions. Since this displacement field is only determined to within an arbitrary rigid body displacement, it is desirable that the solution of the difference equations has the same kind of indeterminacy. This consistency condition is not likely to be fulfilled, when arbitrary finite difference analogs of the differential expressions in the field equations and boundary conditions of the problem are used. A method of establishing consistent finite difference equations is presented and illustrated by examples.

1. INTRODUCTION

THIS paper is concerned with the numerical treatment of problems of plane stress in elastic disks subjected to a given equilibrium system of body forces and edge tractions. For a disk of constant thickness that is free from body forces, the problem is usually formulated in terms of Airy's stress function.† This formulation has the advantage of leading to a standard boundary value problem: the stress function is biharmonic in the region of the disk, and its values as well as those of its normal derivative along the edge of the disk are readily obtained from the given edge tractions. This advantage is lost when the disk has variable thickness or carries body forces. Moreover, the values of the stress function at the nodes of a grid have no direct physical meaning. The numerical determination of the displacements from the stress function is awkward, while that of the stresses involves numerical evaluation of second derivatives and therefore requires use of a rather fine grid for reasonable accuracy. When the problem is formulated in terms of the displacement components at the nodes of a grid, the first difficulty is eliminated and the second is reduced because the determination of the state of stress only requires the numerical evaluation of first derivatives. On the other hand, the displacement field of the given boundary value problem is only determined to within an arbitrary rigid-body displacement. For consistency, the finite difference equations for the problem should therefore be constructed in such a manner that their solution has the same kind of indeterminacy. This consistency condition is not likely to be satisfied when the difference equations are constructed from arbitrary finite difference analogs of the differential expressions in the field equations and boundary conditions of the problem.

To obtain a system of finite difference equations for the displacements that has a unique solution, it is customary to set three suitably chosen displacement components

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† For a numerical example see, for instance [1].

equal to zero and drop the corresponding difference equations from the system. This procedure obscures a lack of consistency of the unabridged system if only one "solution" is obtained. If, however, the procedure is repeated, with another set of three displacement components being set equal to zero, the difference between the displacement fields of the two "solutions" does not correspond to a rigid-body displacement. Accordingly, the two "solutions" are associated with distinct stress fields. This lack of uniqueness of the approximate stress field obtained from inconsistent difference equations is in striking contrast to the uniqueness of the true elastic stress field.

This paper presents a method of establishing consistent difference equations for the displacement components in a thin elastic disk of varying thickness that is subjected to given body forces and edge tractions. A similar treatment of the Neumann problem for the Poisson equation has been given by Friedrichs and Keller [2], who briefly mention the possibility of extending their technique to problems concerning elastic bodies with free edges or surfaces.

2. NOTATIONS AND BASIC RELATIONS

Using rectangular Cartesian coordinates x_1, x_2 in the median plane of the disk, we denote the region covered by the disk by R and its boundary by B, the area element of R by dA, and the arc element of B by ds. To discretize the problem, we cover the x_1, x_2-plane by a triangular grid formed by equidistant* parallels to the coordinate axes and the diagonals of positive slope in the resulting squares (Fig. 1). Following Pólya [3] and Synge [4], we propose to approximate the displacement field of the disk by a linear combination u_j ($j = 1, 2$) of special displacement fields which will be called pyramid fields.

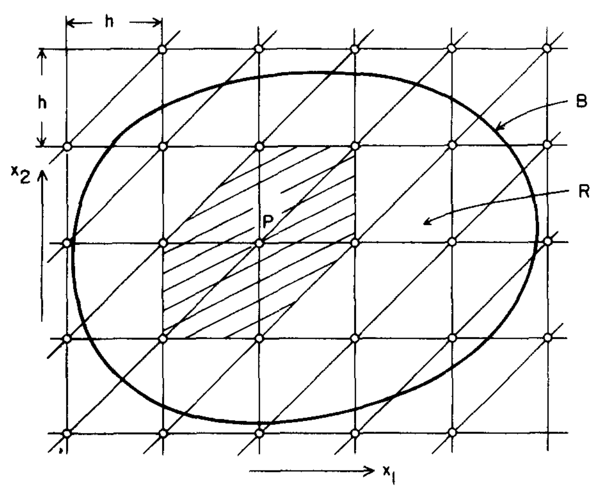

FIG. 1. Region R covered by disk with boundary B. Domain of influence of node P (shaded). Nodes appearing in finite difference equations (circled).

The typical pyramid field is associated with a certain coordinate direction α, ($\alpha = 1, 2$), and a certain node P of the grid; it will be denoted by $u_j^{\alpha P}$. The pyramid field u_j^{1P}, for instance,

* A uniform spacing h of these parallels, while convenient, is not essential.

has $u_2^{1P} \equiv 0$, while u_1^{1P} differs from zero only in the hexagon formed by the nodes that are connected to P by a grid segment. This hexagon, which has been shaded in Fig. 1, will be called the domain of influence of P. The variation of u_1^{1P} over this domain is represented by the pyramidal surface that has zero elevation along the boundary of the domain and unit elevation at P. We denote by $\epsilon_{ij}^{\alpha P}$ the piecewise constant strain field that is associated with the displacement field $u_j^{\alpha P}$, and by $\sigma_{ij}^{\alpha P}$ the piecewise constant stress field that is obtained from $\epsilon_{ij}^{\alpha P}$ by Hooke's law.

For the numerical determination of the displacement field of the disk, only those nodes are relevant whose domain of influence has a common area with the region R. Let S be the set of these nodes, which have been circled in Fig. 1, and let N be their number. Denoting the displacement components of the typical node P in S by U^{1P}, U^{2P}, we consider piecewise linear displacement fields of the form

$$u_j = \sum_{P\in S} \sum_{\alpha=1}^{2} U^{\alpha P} u_j^{\alpha P}. \tag{1}$$

The corresponding piecewise constant fields of strain and stress are

$$\epsilon_{ij} = \sum_{P\in S} \sum_{\alpha=1}^{2} U^{\alpha P} \epsilon_{ij}^{\alpha P}, \qquad \sigma_{ij} = \sum_{Q\in S} \sum_{\beta=1}^{2} U^{\beta Q} \sigma_{ij}^{\beta Q}. \tag{2}$$

Following Ritz, we determine the $2N$ values $U^{\alpha P}$ ($\alpha = 1, 2; P \in S$) in such a manner that the displacement (1) minimizes the potential energy

$$\Pi = \frac{1}{2} \int_R \sigma_{ij} \epsilon_{ij} t \, \mathrm{d}A - \int_R F_j u_j t \, \mathrm{d}A - \int_B T_j u_j t \, \mathrm{d}s, \tag{3}$$

where the summation convention of tensor algebra applies to repeated subscripts, t is the variable thickness of the disk, F_j the body force per unit of volume, and T_j the traction per unit area of the cylindrical edge of the disk. Substituting (1) and (2) into (3), using the reciprocal identity $\epsilon_{ij}^{\alpha P} \sigma_{ij}^{\beta Q} = \sigma_{ij}^{\alpha P} \epsilon_{ij}^{\beta Q}$, and differentiating with respect to a given $U^{\alpha P}$, we obtain as typical minimum condition $\partial \Pi / \partial U^{\alpha P} = 0$ a linear equation for the displacement $U^{\beta Q}$:

$$\sum_{Q\in S} \sum_{\beta=1}^{2} U^{\beta Q} \int_R \sigma_{ij}^{\alpha P} \epsilon_{ij}^{\beta Q} t \, \mathrm{d}A = \int_R F_j u_j^{\alpha P} t \, \mathrm{d}A + \int_B T_j u_j^{\alpha P} t \, \mathrm{d}s, \qquad (\alpha = 1, 2; P \in S). \tag{4}$$

Since there will be an equation of this kind for each of the values $\alpha = 1, 2$ and for each node $P \in S$, we have $2N$ equations for the $2N$ unknown node displacements $U^{\beta Q}$. It will now be shown that these equations are consistent in the sense discussed in Section 1.

3. CONSISTENCY AND ORDER OF APPROXIMATION

Consider an arbitrary rigid-body displacement $\bar{u}_j$ with the displacement components $\bar{U}^{\alpha P}$ at the node P. Since $\bar{u}_1, \bar{u}_2$ are linear functions of the coordinates, we have

$$\bar{u}_j = \sum_{P\in S} \sum_{\alpha=1}^{2} \bar{U}^{\alpha P} u_j^{\alpha P}. \tag{5}$$

The stress field

$$\bar{\sigma}_{ij} = \sum_{P \in S} \sum_{\alpha=1}^{2} \bar{U}^{\alpha P} \sigma_{ij}^{\alpha P} \tag{6}$$

that is associated with the rigid-body displacement (5) vanishes identically.

Each equation of the system (4) corresponds to a certain value of α and to a certain node P. Let us now multiply each equation by the corresponding $\bar{U}^{\alpha P}$ and form the sum of the equations obtained in this manner. In view of equation (6) and the identity $\bar{\sigma}_{ij} \equiv 0$, the sum of the left-hand sides vanishes, which shows that the system (4) is singular.

To investigate whether the equations (4) are consistent, we use (5) to write the sum of the right-hand side of the multiplied equations as

$$\int_R F_j \bar{u}_j t \, \mathrm{d}A + \int_B T_j \bar{u}_j t \, \mathrm{d}s. \tag{7}$$

Since the given body forces and edge tractions are supposed to form an equilibrium system, the principle of virtual displacements requires that the expression (7) vanishes for any rigid-body displacement $\bar{u}_j$. This shows that the equations (4) are consistent.

It is readily shown that, in the absence of body forces and edge tractions, the node displacements of an arbitrary rigid-body displacement satisfy the system (4). To this end, we rewrite (4) by exchanging α with β and P with Q, then replace $U^{\alpha P}$ by the rigid-body displacement $\bar{U}^{\alpha P}$ and set F_j and T_j equal to zero. It then follows from (6) and the identity $\bar{\sigma}_{ij} \equiv 0$ that the resulting equation is satisfied.

For arbitrary body forces and edge tractions, the difference equations (4) will only furnish an approximation to the displacement field of the disk. Let us, however, assume that the given body forces and edge tractions as well as the thickness distribution are compatible with a linear displacement field. To within a rigid-body displacement, the difference equations (4) must then furnish this field exactly, because the class of displacement fields (1), which we have admitted for comparison under the principle of minimum potential energy, includes this field. The system (4) is therefore seen to represent a first-order approximation to the differential equations and natural boundary conditions for the displacements of the disk in the sense that, irrespective of the mesh length h, the method furnishes the exact displacement field when this depends linearly on the coordinates.

4. TYPICAL FINITE DIFFERENCE EQUATIONS

Equation (4) corresponds to a certain value of α and a certain node P. To evaluate the coefficient

$$c^{\alpha P, \beta Q} = \int_R \sigma_{ij}^{\alpha P} \epsilon_{ij}^{\beta Q} t \, \mathrm{d}A \tag{8}$$

of the nodal displacement $U^{\beta Q}$ in this equation, we note that the fields $\sigma_{ij}^{\alpha P}$ and $\epsilon_{ij}^{\beta Q}$ vanish outside the domains of influence of the nodes P and Q, respectively. Thus, $c^{\alpha P, \beta Q} = 0$ whenever the nodes P and Q are not connected by a grid segment. On the other hand, when P and Q are connected in this manner, their domains of influence have two triangular meshes in common, in each of which the fields $\sigma_{ij}^{\alpha P}$ and $\epsilon_{ij}^{\beta Q}$ are constant. If the disk volume corresponding to the typical mesh M is denoted by

$$V_M = \int_M t \, \mathrm{d}A, \tag{9}$$

the contribution of this mesh to $c^{\alpha P, \beta Q}$ is therefore given by $\sigma_{ij}^{\alpha P}(M)\epsilon_{ij}^{\beta Q}(M)V_M$.

In the following, the shear modulus of the disk material will be denoted by G and Poisson's ratio will be assumed to have the value 1/3. The values of $\sigma_{ij}^{\alpha P}(M)\epsilon_{ij}^{\beta Q}(M)$ for the meshes M in the domain of influence of P are given in Figs. 2, 3, and 4. For each relevant

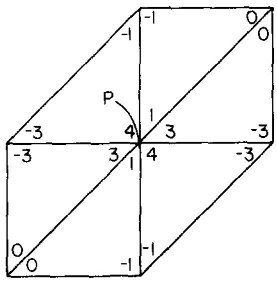

FIG. 2. Values of $(h^2/G)\sigma_{ij}^{1P}(M)\epsilon_{ij}^{1Q}(M)$.

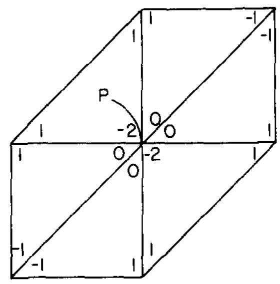

FIG. 3. Values of $(h^2/G)\sigma_{ij}^{1P}(M)\epsilon_{ij}^{2Q}(M)$ or $(h^2/G)\sigma_{ij}^{2P}(M)\epsilon_{ij}^{1Q}(M)$.

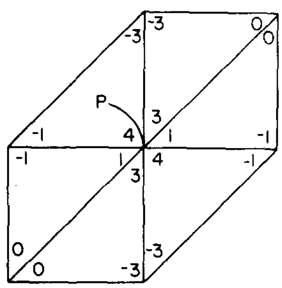

FIG. 4. Values of $(h^2/G)\sigma_{ij}^{2P}(M)\epsilon_{ij}^{2Q}(M)$.

position of Q, the value of $(h^2/G)\sigma_{ij}^{\alpha P}(M)\epsilon_{ij}^{\beta Q}(M)$ for a mesh M that has Q as a vertex is displayed inside this mesh and close to Q. Figure 2 refers to the case $\alpha = \beta = 1$, Fig. 3 to the cases $\alpha = 1$, $\beta = 2$, and $\alpha = 2$, $\beta = 1$, and Fig. 4 to the case $\alpha = \beta = 2$. The manner in which the information in these figures is used to evaluate the coefficients (8) on the left-hand side of the finite difference equation (4) will be illustrated by several examples.

For a disk of the uniform thickness t, the volume associated with each triangular mesh is $V_M = h^2 t/2$. To obtain the values of, say, $(2/Gt)c^{1P,1Q}$ when the entire domain of influence of the node P lies in R, we therefore need only to sum the numbers at each vertex in Fig. 2. Since the values of $(2/Gt)c^{1P,2Q}$ are obtained in the same manner from Fig. 3, the left-hand side of the difference equation (4) for the value $\alpha = 1$ and the node P is therefore represented by Fig. 5. This agrees with the result obtained in [5].

The second example also concerns a disk of the constant thickness t, but the node P is now supposed to lie on a boundary $x_2 =$ const. of the region R. This means that only the numbers in the lower halves of Figs. 2 and 3 are used to obtain the left-hand side of

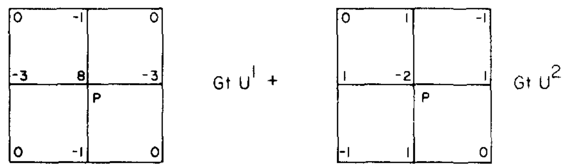

FIG. 5. Left-hand side of difference equation for the value $\alpha = 1$ and the node P (disk of constant thickness t, entire domain of influence of P in R).

the difference equation (4) for the value $\alpha = 1$ and the node P. The result, which is shown in Fig. 6, again agrees with that obtained in [5].

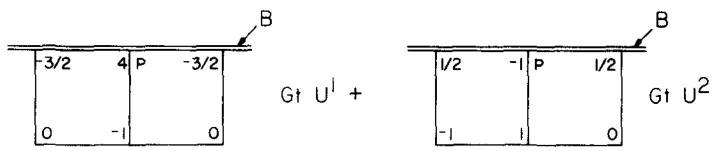

FIG. 6. Left-hand side of difference equation for the value $\alpha = 1$ and the node P on boundary x_2 = const. (disk of constant thickness t).

The third example deals with a curved boundary (Fig. 7), but the disk thickness is still assumed to be constant. Since the disk volume corresponding to the meshes that are labeled M and N in Fig. 7 is only $\pi/4$ times the regular mesh volume $h^2t/2$, the numbers in

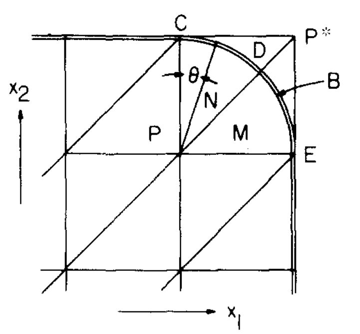

FIG. 7. Quarter-circle boundary (disk of constant thickness).

these meshes in Figs. 2 and 3 must be multiplied by $\pi/4$. This yields the left-hand side shown in Fig. 8. In a similar way, the left-hand side of the difference equation for the value $\alpha = 1$ and the node P^* in Fig. 7 is obtained (see Fig. 9).

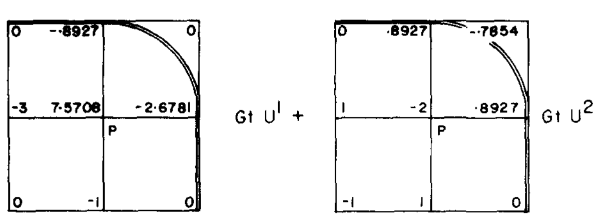

FIG. 8. Left-hand side of difference equation for the value $\alpha = 1$ and the node P at center of quarter-circle boundary (disk of constant thickness t).

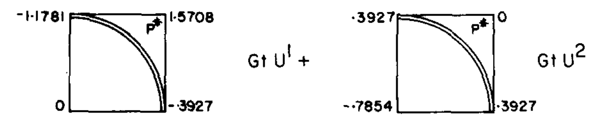

FIG. 9. Left-hand side of difference equation for the value $\alpha = 1$ and the node P^* (disk of constant thickness).

The fourth and last example concerns a disk whose thickness increases linearly with x_1 in the manner indicated at the bottom of Fig. 10. Assuming that the entire domain of influence of the node P lies in the region R, we find the mesh volumes shown in Fig. 10.

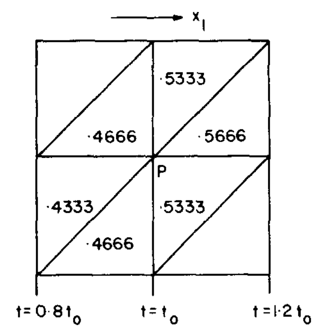

FIG. 10. Mesh volumes for disk with linearly varying thickness (the numbers indicate the quotients of mesh volumes by $h^2 t_0$).

When these volumes are used as factors for the numbers in Figs. 2 and 3, the difference equation indicated in Fig. 11 is obtained.

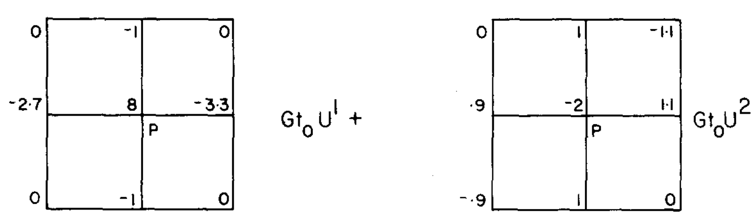

FIG. 11. Left-hand side of difference equation for the value $\alpha = 1$ and the node P (disk of linearly varying thickness shown in Fig. 10).

We next discuss the right-hand side of the difference equation (4). In view of the pyramid character of u_1^{1P} and the vanishing of u_2^{1P}, the contribution of the body force to the right-hand of the difference equation (4) for the value $\alpha = 1$ and the node P may be obtained as follows. For each of the six triangular meshes in the domain of influence of P, regard the quantity, $F_1 t$ as a surface density of mass, and replace this mass distribution over the mesh by point masses at the three vertices that have the same total mass and the same center of gravity as the distributed mass. The sum of the six masses that are allocated to the node P in this way is the value of the first term on the right-hand side of the difference equation (4) for $\alpha = 1$.

If the boundary B intersects a mesh, the contribution of this mesh to the second term on the right-hand side of the difference equation (4) for $\alpha = 1$ may be obtained in the same manner except that $T_1 t$ must now be regarded as a line density of mass, and the mass distribution over the boundary arc in the considered mesh must be replaced by equivalent point masses at the vertices.

To illustrate this procedure, let us determine the contribution of the circular arc CDE in Fig. 7 when the edge tractions on the disk correspond to simple tension of the intensity σ in the x_1 direction. Along the quarter circle CDE, we then have $T_1 = \sigma \sin\theta$, $T_2 = 0$. The arc CD therefore furnishes an equivalent mass at P that is given by

$$\sigma th \int_0^{\pi/4} (1-\cos\theta)\sin\theta \, d\theta = 0{\cdot}0429\sigma th. \tag{10}$$

Similarly, the arc DE furnishes the equivalent mass

$$\sigma th \int_{\pi/4}^{\pi/2} (1-\sin\theta)\sin\theta \, d\theta = 0{\cdot}0644\sigma th, \tag{11}$$

and the total equivalent mass at P is

$$\int T_1 u_1^{1P} t \, ds = 0{\cdot}1073\sigma th. \tag{12}$$

Since the displacements

$$u_1 = \frac{3\sigma}{8G}x_1, \qquad u_2 = -\frac{\sigma}{8G}x_2, \tag{13}$$

which correspond to simple tension of the intensity σ in the x_1 direction are linear in x_1 and x_2, they should exactly fulfill the difference equation the left-hand side of which is shown in Fig. 8 while the right-hand side is given by (12). We leave it to the reader to verify this.

5. CONCLUDING REMARK

Aside from furnishing consistent difference equations, the method outlined above is advantageous when the boundary B does not entirely consist of grid segments. Indeed, the multiplication of the values in Figs. 2 through 4 by the appropriate mesh volumina is a very simple procedure compared with the usual elimination of the displacement components at grid points outside the boundary B by the use of boundary conditions (see, for instance [6]). Similarly, the presence of body forces or variations in the thickness of the disk does not lead to the complications that are encountered in the usual derivation of finite difference equations for the displacement field of the disk.

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Résumé—Ce document s'occupe de l'établissement d'équations consistantes de différences finies pour le champ de déplacement d'un disque élastique fin, d'épaisseur variable sous une force de corps donnée et une traction des bords. Puisque le champ de déplacement n'est déterminé qu'à patir d'un déplacement de corps rigide et arbitraire, il est souhaitable que la solution des équations de difference aient le même genre d'interminance. Cette condition de consistance n'a pas beaucoup de chances d'être réalisée, lorsque des differences finies arbitraires analogues aux expressions differentielles dans les équations de champ et les conditions de limite du problème sont employées. Une méthode pour établir des équations consistantes de differences finies est présentée et illustrée d'exemples.

Zusammenfassung—Diese Arbeit befasst sich mit der Festlegung von konsistenten endlichen Differenzgleichungen für das Verschiebungsfeld einer dünnen elastischen Scheibe veränderbarer Dicke bei gegebenen Körperkräften und Kantenzügen. Da das Verschiebungsfeld nur innerhalb einer willkürlichen Starrkörper-Verschiebung bestimmt wird, ist es erwünscht, dass man die Lösung der Differenzgleichungen mit denselben Unbestimmtheiten ausdrückt. Diese Konsistenzbedingung kann kaum erfüllt werden, wenn willkührliche endliche Differenzanaloge der Differenzialausdrücke in den Feldgleichungen und Kantenbedingungen verwendet werden. Eine Methode wird gegeben zur Erzielung konsistenter endlicher Differenzgleichungen, Beispiele werden auch gegeben.

Абстракт—Статья касается установления уравнений совместимой конечной разности для поля перемещения тонкого эластического диска переменной толщины при данных силах тела и тяге края. Так как это поле перемещения определяется только в пределах произвольного перемещения твёрдого тела, желательно, чтобы решение уравнений в конечных разностях обладало такого же рода неопределённостью. Это условие совместимости не выполняется, когда применяются аналоги произвольной конечной разницы дифференциальных выражений в уравнениях поля и в условиях границы проблемы. Предлагается метод установления совместимых уравнений конечной разницы и поясняется примерами.